Q1. a. $A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}$

$P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & -6 \\ 0 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$

$Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & 2i \\ 2i & 2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

b) Solving the quadratic eq^n $x^2+2x+2=0$.

$x = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

Let $\alpha = -1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\, e^{3\pi i/4}$

$\beta = -1-i = \sqrt{2}\left(\cos\frac{5\pi i}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\, e^{5\pi i/4}$

$\alpha^n \cdot \beta^n = 2^{n/2}\, 2^{n/2}\, e^{3n\pi i/4}\, e^{n5\pi i/4}$

$= 2^n e^{i8n\pi/4}$

$= 2^n e^{4n\pi i} = 2^n[\cos 4n\pi + i\sin 4n\pi]$

$= 2^n$

c) $f(x) = \log(1+e^x)$ $\qquad f'(x) = \frac{e^x}{1+e^x}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \cancel{\frac{1}{2^2}}$ $\qquad f''(0) = \frac{1}{2^2}$

$= \frac{e^x}{(1+e^x)^2}$

$f'''(x) = \frac{(1+e^x)^2 e^x - e^x 2(1+e^x)e^x}{(1+e^x)^4}$

$= \frac{e^x - e^{2x}}{(1+e^x)^3}$ $\qquad f'''(0) = 0$

$$f^{iv}(x) = \frac{d}{dx}\left[\frac{e^x - e^{2x}}{(1+e^x)^3}\right]$$

$$= \frac{(1+e^x)^3(e^x - 2e^{2x}) - 3(1+e^x)^2 e^x(e^x - e^{2x})}{(1+e^x)^6}$$

$$= \frac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^4}$$

$$f^{iv}(0) = \frac{2(1-2) - 3(1-1)}{2^4} = \frac{-1}{2^3}.$$

Hence by Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$$

d) If $z = f(x,y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^{v}$ then p.t

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Sol^n $\quad z \to f(x,y) \to u, v.$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u} \quad (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^{v}. \quad (2)$$

Subtracting 1 & 2.

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-u})\frac{\partial z}{\partial x} - (e^{-u} - e^{u})\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

e) Find n^{th} derivative of $\dfrac{x^2}{(x-1)^2(x+2)}$

Sol^n Consider $\dfrac{x^2}{(x-1)^2(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{(x-1)^2} + \dfrac{c}{x+2}$

$\Rightarrow x^2 = a(x-1)(x+2) + b(x+2) + c(x-1)^2$

Put $x = 1$ $\qquad b = 1/3$

$x = -2$ $\qquad 4 = 9c \qquad c = 4/9$

$x = 0$ $\qquad 0 = -2a + 2b + c \qquad a = 5/9$

$$\therefore\ y = \frac{5}{9}\frac{1}{x-1} + \frac{1}{3(x-1)^2} + \frac{4}{9(x+2)}$$

$$y_n = \frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{3}\frac{(n+1)!(-1)^n}{(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$= \frac{(-1)^n n!}{9}\left[\frac{5}{(x-1)^{n+1}} + \frac{3(n+1)}{(x-1)^{n+2}} + \frac{4}{(x+2)^{n+1}}\right].$$

f) Using Newton-Raphson method for the equation $x^3-5x+3=0$, find the roots starting with $x_0=1$ correct upto four decimal places.

<u>Sol^n</u>

$$f(x) = x^3-5x+3 \qquad f'(x) = 3x^2-5$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$= x_n - \frac{x_n^3-5x_n+3}{3x_n^2-5}$$

$$= \frac{2x_n^2 - 3}{3x_n^2-5} \qquad = 3x_n^3 - \cancel{5x_n} - x_n^3 + \cancel{5x_n} - 3$$

When $x_0 = 1$ $\quad x_1 = 0.5$

$x_1 = 0.5 \quad x_2 = .5 - \dfrac{(.5)^3 - 5(.5)+3}{3(.5)^2-5} = 0.6471$

$x_2 = .6471 \quad x_3 = .6471 - \dfrac{(.6471)^3 - 5(.6471)+3}{3(.6471)^2-5}$

$= 0.6566$

$x_3 = 0.6566 \quad x_4 = .6566.$

$\therefore\ x = 0.6566.$

Q

Q2. a. $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

$R_2 - R_1$, $R_3 - 5R_1$ $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$ — 2

$C_2 + C_1$, $C_3 - 3C_1$, $C_4 - 6C_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$ — 4

$R_3 - 2R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ — 6

$R_2/4$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/2 & -5/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 7

$C_3 + \frac{3}{2}C_2$, $C_4 + \frac{5}{2}C_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 8

C_{34} $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ Rank = 3. 10.

b) Using De Moivre's Theorem, prove that
$\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$

Sol^n Let $x = \cos\theta + i\sin\theta$ $\quad 1/x = \cos\theta - i\sin\theta$

Adding $x + \frac{1}{x} = 2\cos\theta$

Subtracting

$x - \frac{1}{x} = 2i\sin\theta$ (2)

$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6$

$= x^6 + 6x^5\frac{1}{x} + 15x^4\frac{1}{x^2} + 20x^3\frac{1}{x^3} + 15x^2\frac{1}{x^4} + 6x\frac{1}{x^5} + \frac{1}{x^6}$

$= x^6 + 6x^4 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$ (4)

—(1)

$(2i\sin\theta)^6 = x^6 - 6x^4 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$ —

$+2^6\sin^6\theta = -x^6 + 6x^4 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}$ — 6.

(2)

Adding 1 & 2

$2^6(\cos^6\theta + \sin^6\theta) = 12x^4 + 40 + 12\frac{1}{x^4}$ 8

$= \frac{1}{16}\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$ 9

$= \frac{1}{16}[3\cos 4\theta + 10]$ 10.

Q2.c. If $\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$, prove that $\alpha=\frac{n\pi}{2}+\frac{\pi}{4}$, $\beta=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Sol^n

We have

$\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$

$\tan(\alpha-i\beta)=\cos\theta-i\sin\theta$ 1

$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$

$= \dfrac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$ 2.

$= \dfrac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}$

$= \dfrac{2\cos\theta}{0}$ 3

$2\alpha = \tan^{-1}(\infty)$

$= \pi/2$ 4

$2\alpha = n\pi+\pi/2$

$\alpha = \frac{n\pi}{2}+\frac{\pi}{4}$ 5

Also $\tan(2i\beta)=\tan[(\alpha+i\beta)-(\alpha-i\beta)]$

$= \dfrac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$ 6

$= \dfrac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)}$

$= i\sin\theta$ 7

$i\tanh 2\beta = i\sin\theta$

$\tanh 2\beta = \sin\theta$

$2\beta = \tanh^{-1}(\sin\theta)$ 8.

$= \frac{1}{2}\log\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)$

$= \frac{1}{2}\log\left[\dfrac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right]^2$ 9

$\beta = \frac{1}{2}\log\left[\dfrac{1+\tan\theta/2}{1-\tan\theta/2}\right] =$

$= \frac{1}{2}\log\tan\left(\pi/4+\theta/2\right)$ 10.

Q2.d. If $u = \sin^{-1}(x^2+y^2)^{1/5}$ prove that

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{5}\tan u$

ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{2}{25}\tan u[2\tan^2 u - 3]$

Soln Here u is not hom functn in x & y.

let $z = \sin u = (x^2+y^2)^{1/5}$

Put $x = Xt$ $y = Yt$

$z = \sin u = t^{2/5}(X^2+Y^2)^{2/5}$

$\therefore$ $z = f(u) = \sin u$ is a hom functn of degree $2/5$

2

i) Hence by cor of Eulers theorem

$$x u_x + y u_y = \frac{n f(u)}{f'(u)} = \frac{2}{5}\frac{\sin u}{\cos u}$$

$$= 2/5 \tan u.$$ 4.

ii) By Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where $g(u) = \frac{n f(u)}{f'(u)}$ 6

$$= \frac{2}{5}\tan u$$

$$g'(u) = \frac{2}{5}\sec^2 u$$

$$g(u)[g'(u)-1] = RHS = \frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u - 1\right]$$

$$= \frac{2}{25}\tan u[2\sec^2 u - 5]$$ 8.

$$= \frac{2}{25}\tan u[2(1+\tan^2 u) - 5]$$

$$= \frac{2}{25}\tan u[2\tan^2 u - 3]$$ 10.

Q2. e. If $x = \sin \frac{1}{m}\log y$ then prove that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$.

Sol^n

$$x = \sin\left(\frac{1}{m}\log y\right)$$

$$\sin^{-1}x = \frac{1}{m}\log y$$

$$\log y = m\sin^{-1}x$$

$$y = e^{m\sin^{-1}x}$$ 2.

Diff wrt 'x'

$$y_1 = e^{m\sin^{-1}x}\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = my \quad \text{———} \; (1)$$ 4

Again diff wrt 'x'

$$\sqrt{1-x^2}\,y_2 - \frac{xy_1}{\sqrt{1-x^2}} = my_1$$

From eq^n (1)

$$\sqrt{1-x^2}\,y_2 - \frac{xy_1}{\sqrt{1-x^2}} = \frac{m^2y}{\sqrt{1-x^2}}$$

$$(1-x^2)y_2 - xy_1 - m^2y = 0$$ (6)

Applying Leibnitz theorem on each term

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n$$
$$-[xy_{n+1} + ny_n] - m^2y_n = 0.$$ 8.

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$ 10.

Q2.f. Solve the following eq^n by Gauss-Seidel method upto four iterations.

$4x - 2y - z = 40$, $x - 6y + 2z = -28$, $x - 2y + 12z = -86$.

Sol^n Eq^n can be rewrite as.

$x = \frac{1}{4}[40 + 2y + z]$ (1)

$y = \frac{1}{6}[28 + x + 2z]$ (2)

$z = \frac{1}{12}[-86 - x + 2y]$ (3)

(2).

1st iteration

Put $y = z = 0$ in eq^n (1)

$x_1 = \frac{1}{4}(40) = 10$

Put $x = 10$, $z = 0$ in eq^n (2)

$y = \frac{1}{6}[28 + 10 + 0] = 6.3333$

Put $x = 10$, $y = 6.3333$ in eq^n (3)

$z = \frac{1}{12}[-86 - 10 + 2(6.3333)] = -6.945$

Iteration	x	y	z
1	10	6.3333	-6.945
2	11.4306	4.2569	-7.4097
3	10.2760	3.9094	-7.3714
4	10.1118	3.8948	-7.3602

$\therefore$ $x = 10.11$ $y = 3.89$ $z = -7.36$

Q 3.a Test for consistency and solve

$x+y+z=3, \quad x+2y+3z=4, \quad x+4y+9z=6$

Solⁿ Writing in matrix form

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \qquad 1$$

Augmented matrix

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

$$\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right] \qquad 2$$

$$R_3 - 3R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right] \qquad 3$$

Rank of (A) = Rank of $(AB) = 3$
$= n$ 4

Solution is consistent and unique

$x+y+z = 3$ $\qquad x = 2$

$y+2z = 1$ $\qquad y = 1$ 5.

$2z = 0$ $\qquad z = 0$

b) Seperate into real and imagary part of $\cos^{-1}(i\theta)$.

Solⁿ Let $\cos^{-1}(i\theta) = x+iy$ 1

$\cos(x+iy) = i\theta$

$\cos x \cos iy - \sin x \sin iy = i\theta$

$\cos x \cosh y - i \sin x \sinh y = i\theta$ 2

Equating real and imag part on both sides

$\cos x \cosh y = 0$ $\qquad -\sin x \sinh y = \theta$ 3

$\cos x = 0$ $\qquad \sin \frac{\pi}{2} \sinh y = -\theta$ 4

$x = \pi/2$

$\sinh y = -\theta$

$y = \sinh^{-1}(-\theta)$

$y = -\log(\theta + \sqrt{\theta^2+1})$. 5

$\cos^{-1} i\theta = \frac{\pi}{2} - i\log(\theta + \sqrt{\theta^2+1})$

Q3. c. If $u = (1-2xy+y^2)^{-1/2}$ prove that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3.$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y) = yu^3 \quad 1$$

$$x\frac{\partial u}{\partial x} = xyu^3 \quad 2$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y) \quad 3$$

$$\frac{\partial u}{\partial y} = (x-y)u^3$$

$$y\frac{\partial u}{\partial y} = (xy-y^2)u^3. \quad 4$$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = (xy - xy + y^2)u^3 = y^2u^3 \quad 5.$$

d) Examine the function $f(x,y) = y^2+4xy+3x^2+x^3$ for extreme values.

$f(x,y) = y^2+4xy+3x^2+x^3$

$f_x = \frac{\partial f}{\partial x} = 4y+6x+3x^2$

$f_y = \frac{\partial f}{\partial y} = 2y+4x$

$f_{xx} = 6+6x = r$

$f_{yy} = 2 = t$

$f_{xy} = 4 = s.$

Solving $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

$4y+6x+3x^2 = 0$ & $2y+4x = 0$.

Put $y = -2x$ in eq 1.

$-8x+6x+3x^2 = 0.$

$3x^2-2x = 0.$

$x(3x-2) = 0.$

$x = 0 \quad x = 2/3$

$y = 0 \quad y = -4/3$

Stationary pts. $(0,0)$, $(2/3, -4/3)$ - 3.

1) When $x = 0$ $y = 0$.

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = 4.$

$rt - s^2 = 12-16 < 0.$

neither max or min exist

— 4

2) When $x = 2/3, y = -4/3$.

$r = 10 \quad s = 4 \quad t = 2$

$rt - s^2 = 20-16 > 0.$

$\therefore r = 10 > 0$

$f(x,y)$ min exist

$f_{min}(2/3, -4/3) = -4/27.$ — 5

e) Prove that $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

Sol^n $\log(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \cdots$

$\therefore$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\log(1+\sin x) = \left(x - \frac{x^3}{3!} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2$$
$$+ \frac{1}{3}\left(x - \frac{x^3}{3!} + \cdots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} - \cdots\right)^4 + \cdots$$
$$= x - \frac{x^2}{2} + x^3\left(-\frac{1}{6} + \frac{1}{3}\right) + x^4\left(\frac{1}{6} - \frac{1}{4}\right) + \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

f) Fit a st. line for the following data

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9.

Sol^n St. line y on x; x as ind var.

$y = a + bx$ —— (1)

Normal eq^n

$\Sigma y = na + b\Sigma x$

$\Sigma xy = n\Sigma x + b\Sigma x^2$

$\Sigma y = 40$ $\Sigma x = 56$ $\Sigma xy = 364$ $\Sigma x^2 = 524$

$n = 8$.

$40 = 8a + 56b$.

$364 = 56a + 524b$.

$a = +0.545$ $b = +0.636$

$y = +0.545 + 0.636x$